STABILITY AND BUCKLING OF HETEROGENEOUS COLUMNS BEYOND THE ELASTICITY LIMIT

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It was noted in [1] that the behavior of heterogeneous structures (bimetals, three-ply, and reinforced materials) beyond the elasticity limit has certain singularities which preclude the use of the concept of prolonged loading (the Shanley concept) in problems of their inelastic instability. The compression of such structures beyond the elastic limit is characterized by the appearance of several critical loads of various types depending on the properties of materials used in such a structure and on its geometrical parameters. Nevertheless, one may posultate that critical loads of the Shanley type (as defined in the concept of prolonged loading) are not realized in heterogeneous structures compressed beyond the elastic limit except in the case of the "proportional" tensile stress-strain diagrams of the materials of a laminated structure.

In the general case the critical load (in the usual meaning of the load producing initial change from one form of motion to another) of a heterogeneous structure should be defined as the load corresponding to the yield point. For a certain class of geometrical and physical parameters of the structure this load determines its supporting capacity; for the remaining parameters it may serve as the lower limit of supporting capacity loads. The term "supporting capacity loads" denotes here the maximum loads which can be carried by the structure compressed beyond the elastic limit. These general assertions are proved in this article for the example of Shanley's heterogeneous idealized column.

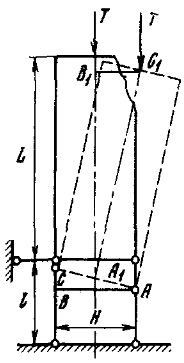

Fig. 1

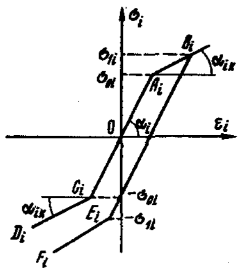

Fig. 2

1. Consider the compression of a Shanley idealized column [2] (Fig. 1) whose segments are made of different elastic-plastic materials with linear strain-hardening characteristics (Fig. 2).

Here L is the length of the rigid part of the column, l is the length of its deformed part, F_i are areas of the transverse cross section of the supporting segments, H_i is the distance from the middle of a segment to the line of action of the applied compressive force T, σ_{0i} are the proportionality limits, E_i are the elastic moduli, and E_{ki} are the tangent moduli of the first and second segments of the column (i = 1, 2). It is assumed that each material behaves in the same way in tension and compression. The line of action of the applied compressive load is chosen so as to ensure that the column remains in a moment-free unbuckled state until the Euler critical load

$$T_{Eu} = \sigma_{02} F_2 t_{Eu} \quad t_{Eu} = \frac{\omega^2 e_2}{1+\alpha f} \tag{1.1}$$

$$\left(\omega^2 = \frac{H^2}{Ll}, \quad \alpha = \frac{e_2}{e_1}, \quad f = \frac{F_2}{F_1}, \quad e_i = \frac{E_i}{\sigma_{02}}\right)$$

is reached. Then

$$h_1 = \frac{\alpha f}{1+\alpha f}, \quad h_2 = \frac{1}{1+\alpha f}, \quad h_i = \frac{H_i}{H} \quad H_1 + H_2 = H.$$

An elastically stable column remains straight until at least one of the segments passes to the plastic state. Let the segment materials be such that

$$\frac{1}{e_2} < \frac{s_{01}}{e_1} \quad \left(s_{0i} = \frac{\sigma_{0i}}{\sigma_{02}}\right). \tag{1.2}$$

Then the second segment passes to the plastic state at a compressive load

$$T_1 = \sigma_{02} F_2 t_1, \quad t_1 = \frac{1 + \alpha f}{\alpha f}. \tag{1.3}$$

The load t_1 will be called the first critical load. Henceforth our considerations will be confined to columns for which $t_1 < t_{Eu}$ or

$$\omega^2 > \frac{(1 + \alpha f)^2}{\alpha f e_2}. \tag{1.4}$$

At $t > t_1$, in accordance with the tensile stress-strain diagrams (Fig. 2), the relation between stresses σ_i and strains ε_i (not taking into account the Bauschinger effect) has the form

$$\varepsilon_i = s_i / e_i + k_i \mu_i [s_i - \operatorname{sign}(s_i) s_{0i}] + m_i \mu_i [s_i - \operatorname{sign}(s_i) s_{1i}] + n_i \mu_i (s_{1i} - s_{0i}) \tag{1.5}$$

$$s_i = \frac{\sigma_i}{\sigma_{02}}, \quad \mu_i = \frac{(e_i - e_{ki})}{e_i e_{ki}}, \quad e_{ki} = \frac{E_{ki}}{\sigma_{02}}, \quad s_{1i} = \frac{\sigma_{1i}}{\sigma_{02}}$$

modes $A_i C_i$ $k_i = m_i = n_i = 0$

modes $A_i B_i$, $C_i D_i$ $k_i = 1, \quad m_i = n_i = 0$

modes $B_i E_i$ $n_i = 1, \quad k_i = m_i = 0$

modes $E_i F_i$ $m_i = n_i = 1, \; k_i = 0.$

Here σ_{1i} denote the yield points of the segment materials in compression (after strain-hardening). Henceforth the compressive stresses are regarded as positive.

The equation of consistency of deformation, which can be obtained by considering the similarity of triangles ABC and $A_1B_1C_1$ (Fig. 1) and regarding the deflection W as small in comparison with L, has the form

$$\varepsilon_2 - \varepsilon_1 = \omega^2 w, \qquad \varepsilon_i = (l_i - l) / l, \qquad w = W / H. \tag{1.6}$$

Here l_i denotes the lengths of segments after deformation. Using the equation of equilibrium of the column

$$s_1 = ft(h_2 - w), \quad s_2 = t(h_1 + w) \quad (T = \sigma_{02} F_2 t) \tag{1.7}$$

and Eqs. (1.5) and (1.6), we obtain

$$w = \frac{c + at}{\omega^2 - bt} \tag{1.8}$$

$$a = (k_2 + m_2)\mu_2 h_1 - (k_1 + m_1)\mu_1 f, \quad b = \beta + (k_2 + m_2)\mu_2 + (k_1 + m_1)\mu_1 f$$
$$c = k_1 \operatorname{sign}(s_1) s_{01}\mu_1 - k_2 \operatorname{sign}(s_2)\mu_2 + m_1 \operatorname{sign}(s_1) s_{11}\mu_1 - m_2 \operatorname{sign}(s_2) s_{12}\mu_2 +$$
$$+ n_2\mu_2(s_{12} - 1) - n_1\mu_1(s_{11} - s_{01}), \quad \beta = (1 + \alpha f) / e_2.$$

From (1.8) at $k_i = m_i = n_i = 0$ we obtain Euler's critical load (1.1) for a heterogeneous idealized column. In the case in which inequalities (1.2) and (1.4) are satisfied, we take in (1.8) $k_2 = 1$, $k_1 = m_i = n_i = 0$.

Then at $t > t_1$ we obtain

$$w = \frac{A(t - t_1)}{t_* - t}, \quad A = \frac{f h_2 (e_2 - e_{k2})}{\beta_1}, \quad \beta_1 = e_1 + e_{k2} f, \quad t_* = \frac{\omega^2 e_1 e_{k2}}{\beta_1}. \tag{1.9}$$

From the equation of equilibrium (1.7), using Eq. (1.9), we obtain

$$s_1'(t) = \frac{e_1 f [(t - t_*)^2 - B]}{\beta_1 (t_* - t)^2}, \quad s_2'(t) = \frac{e_{k2} f [(t - t_*)^2 + e_1 B / e_{k2} f]}{\beta_1 (t_* - t)^2},$$
$$B = \frac{f t_* (e_2 - e_{k2})(t_* - t_1)}{\beta e_1 e_2}, \quad s_1'(t_1) = \frac{f(t_{Eu} e_{k2} - e_2 t_1)}{(t_* - t_1)\beta_1 \alpha}, \quad s_2'(t_1) = \frac{e_{k2} f (t_{Eu} - t_1)}{(t_* - t_1)\beta_1}. \tag{1.10}$$

The prime denotes a derivative with respect to t. Depending on material properties and geometrical parameters of the column, at $t = t_1 < t_{Eu}$ the following cases are possible:

$$a)\ t_* > t_1,\ t_{Eu}e_{k2} > t_1 e_2. \tag{1.11}$$

Then $s_1'(t_1) > 0$, $s_2'(t_1) > 0$, i.e., overloading takes place in both segments; in the second segment it occurs at all the $t > t_1$ and in the first segment at $t_1 < t < t_m$

$$t_m = t_* - \sqrt{B}\ \ (t_1 \leqslant t_m \leqslant t_*). \tag{1.12}$$

Taking in (1.7) $s_1 = s_{01}$ and using (1.9), we obtain

$$t_{10}^+ = p - \sqrt{p^2 - q}, \qquad p = \frac{s_{01} + fh_2t_* + Aft_1}{2(h_2 + A)f}, \qquad q = \frac{s_{01}t_*}{(h_2 + A)f}.$$

The load t_{10}^+ corresponds to reaching the yield point during the compression of the first segment. Let us first consider materials and column parameters for which

$$t_m \leqslant t_{10}^+. \tag{1.13}$$

Then, if inequality (1.13) is satisfied, it follows from (1.10) that at $t_1 \leq t \leq t_m$ elastic overloading takes place in the first segment, elastic unloading taking place at $t > t_m$ until stress σ_1 becomes equal to the tensile yield point. For the force t^-_{10}, which corresponds to stress $\sigma_1 = -\sigma_{10}$, from (1.7) and (1.10) we obtain

$$t_{10} = r + \sqrt{r^2 + q}, \qquad r = p - q / t_*.$$

Using the first of inequalities (1.11), it is easy to prove the validity of inequalities

$$t_m \leqslant t_{10}^- \leqslant t_* \leqslant t_{Eu}. \tag{1.14}$$

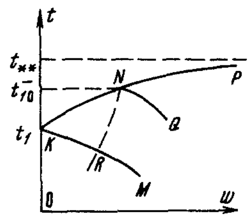

Fig. 3

From (1.10) it follows that, if inequality (1.14) is satisfied, the derivative $s_1'(t_{10}^-)$ is negative, since in this case

$$(t_{10}^- - t_* - \sqrt{B})\,(t_{10}^- - t_* + \sqrt{B}) < 0. \tag{1.15}$$

In the case under consideration, when inequalities (1.11) and (1.13) are satisfied, the dependence of deflection on load at $t \leq t_{10}^-$ is described by (1.9) and represented by curve KN in Fig. 3.

Consider the behavior of the column under loads $t > t_{10}^-$, assuming that inequalities (1.11) and (1.13) are satisfied; in these circumstances the first segment passes into a plastic state in tension (mode D_1C_1 in Fig. 2) and the second segment remains in the plastic state in compression (mode A_2B_2).

Using (1.5) and (1.8), we obtain the following expression for the deflection:

$$w = w_0^- + \frac{D^-(t - t_{10}^-)}{t_{**} - t} \tag{1.16}$$

$$D^- = \frac{f\,[(e_2 - e_{k2})\beta_2(t_* - t_1) - e_{k2}e_2(e_1 - e_{k1})\beta(t_* - t_{10}^-)}{e_2\beta\beta_1\beta_2(t_* - t_{10}^-)}$$

$$t_{**} = \frac{\omega^2 e_{k1}e_{k2}}{\beta_2}, \qquad \beta_2 = e_{k1} + e_{k2}f, \qquad w_0^- = \frac{A(t_{10}^- - t_1)}{t_* - t_{10}^-}.$$

Using (1.16), from (1.7) we obtain

$$s_1'(t) = \frac{e_{k1}f\,[(t - t_{**})^2 - R^-]}{\beta_2(t_{**} - t)^2}, \qquad R^- = \frac{D^-}{e_{k1}}\beta_2 t_{**}(t_{**} - t_{10}^-)$$

$$s_2'(t) = \frac{e_{k2}f\,(t - t_{**})^2 + R^-e_{k1}}{\beta_2(t_{**} - t)^2}, \qquad s_1'(t_{10}^-) = \frac{e_{k1}f\,[(t_{10}^- - t_*)^2 - B]}{\beta_2(t_* - t_{10}^-)(t_{**} - t_{10}^-)} \tag{1.17}$$

$$s_2'(t_{10}^-) = \frac{\beta_2(t_* - t_{10}^-)(t_{**} - t_{10}^-) - e_{k1}[(t_{10}^- - t_*)^2 - B]}{\beta_2(t_* - t_{10}^-)(t_{**} - t_{10}^-)}$$

Consider two possible variants of the case in question.

a_1) Let the segment materials and column parameters be such that in addition to inequalities (1.11) and (1.13) the following inequality is satisfied:

$$t_{**} > t_{10}^- . \tag{1.18}$$

Then, on the basis of (1.14) and (1.15), we have $s_1'(t_{10}^-) < 0$, $s_2'(t_{10}^-) > 0$. The first of these inequalities is equivalent, in accordance with (1.17), to inequalities

$$t_{**} - \sqrt{R^-} < t_{10}^- < t_{**} + \sqrt{R^-}, \quad R^- > 0. \tag{1.19}$$

Then for any t in the interval $t_{10}^- < t < t_{**} + \sqrt{R^-}$ we have $s_1'(t) < 0$, $s_2'(t) > 0$. Consequently, solution (1.16) can be used in the entire above-cited range of t. In accordance with (1.19) the deflection is a monotonically increasing function of the force t, and at $t \to t_{**}$, $w \to \infty$. Thus, if $t > t_1$ and if inequalities (1.11), (1.13), and (1.18) are satisfied, the deflection increases from zero to infinity when the load varies in the interval $t_1 \leq t < t_{**}$.

The corresponding curve KNP is shown schematically in Fig. 3. The load t_{**} will be called the second critical load.

a_2) If the segment materials and column parameters are such that in addition to inequalities (1.11) and (1.13) the following inequalities are satisfied:

$$t_{**} < t_{10}^- \tag{1.20}$$

then assuming that both segments are in the plastic state, we have $s_1'(t_{10}^-) > 0$. From (1.7), (1.9), (1.12), (1.14), (1.15), and (1.20) it follows then that for $\Delta t = t - t_{10}^- > 0$

$$s_1 + s_{01} = \frac{e_{k1} f \Delta t\,[(t_{10}^- - t_*)^2 - B - \Delta t\,(t_* - t_{10}^-)]}{\beta_2\,(t_* - t_{10}^-)\,(t_{**} - t)} > 0.$$

The latter means that the postulated law of plastic deformation in the first segment is violated. The assumption that at $t > t_{10}^-$ the stress in the first segment remains constant and equal to the tensile yield point also leads to the violation of equations of equilibrium and deformation consistency. Let us now postulate that at $t > t_{10}^-$ the second segment is overloaded in the plastic state (portion A_2B_2) and the first segment is in the elastic state (portion A_1C_1, Fig. 2). Then, as can be easily shown, one arrives at a contradiction with inequalities (1.15) or, equivalently, with the law of deformation of the first segment in portion A_1C_1. The assumption that in the second segment, starting from the load t_{10}^-, elastic unloading takes place along a certain line B_2E_2 and that the first segment is strained in tension along line C_1D_1 evidently contradicts inequality $t > t_{10}^-$. A case is also possible in which at $t > t_{10}^-$ the second segment is unloaded along B_2E_2 and the stress in the first segment corresponds to the portion A_1C_1; in principle it is possible for the load to increase above t_{10}^- owing to different rates of decrease in stress moduli in the two segments. In this case from (1.5) and (1.8) we obtain for the deflection

$$w = \frac{w_0^-\,(t_{\mathrm{Eu}} - t_{10}^-)}{(t_{\mathrm{Eu}} - t)}. \tag{1.21}$$

Substituting this value into (1.7) and using (1.14), we obtain

$$s_2'(t_{10}^-) = h_1 + \frac{A t_{\mathrm{Eu}} (t_{10}^- - t_1)}{(t_* - t_{10}^-)\,(t_{\mathrm{Eu}} - t_{10}^-)} > 0$$

which contradicts the starting assumption about unloading in the second segment at $t = t_{10}^-$.

Thus, in the case of columns for which inequalities (1.11), (1.13), and (1.20) are satisfied, equilibrium states at $t > t_{10}^-$ cannot exist. An additional increase in the deflection is possible only when the load is reduced. The dependence of the deflection on load is in this case represented in Fig. 3 by curve KNQ. Branch KN corresponds to solution (1.9), solution (1.16) corresponds to branch NQ at $e_{k1}^0 \leq e_{k1} \leq e_{k1}^1$, and solution

$$w = w_0^- + D_y^- \frac{(t - t_{01}^-)}{(t_{***} - t)}, \qquad D_y^- = \frac{-\mu_1 h_2 f + (\beta + \mu_1 f)\, w_0^-}{\beta + \mu_1 f}$$
$$e_{k1}^0 = \frac{e_1 e_{k2} f t_{10}^- (t_* - t_{10}^-)}{e_{k2} f t_* (t_* - t_{10}^-) + e_1 B}, \qquad e_{k1} = \frac{e_1 e_{k2} f t_{10}^-}{t_* \beta_1 - e_1 t_{10}^-}, \qquad t_{***} = \frac{\omega^2}{\beta + \mu_1 f} \tag{1.22}$$

at $0 \leq e_{k1} \leq e_{k1}^0$; the latter corresponds to mode D_1C_1 in the first segment and mode B_2E_2 in the second segment.

The load t_{10}^{-} will be called the third critical load. For columns of the kind under consideration this load characterizes their supporting capacity.

It should be noted that the load t_{10}^{-} (and similar loads) constitutes a critical load in the usual bifurcational sense since – when the load is reduced after reaching t_{10}^{-} – two equilibrium curves are possible: curve NQ and curve NR described by solution (1.21) which corresponds to elastic unloading in both segments. As above, it is easy to show that when inequalities (1.18) and $t_* > t_1$ are satisfied, we have

$$t_{Eu} e_{k2} \leqslant t_1 e_2 \tag{1.23}$$

and the behavior of columns corresponds to the branch KNP described by (1.9) and (1.16), while if inequalities (1.20) and (1.23) are satisfied this behavior corresponds to the curve KNQ described by (1.21) or (1.22) with a critical load t_{10}^{-}. If, however, inequalities $t_* < t_1$, $t_{Eu} e_k^2 \leq t_1 e_2$ are satisfied, the behavior of a column corresponds to the curve KM described by (1.9); the supporting capacity of such columns is determined by their first critical load (1.3). The results of analysis carried out for other cases are given below.

b) $t_* > t_1,\quad t_{Eu} e_{k^2} > t_1 e_2,\quad t_m > t_{10}^+,\quad e_{k1} > e_{k1}^*$

$$e_{k1}^* = \frac{e_1 e_{k2} f t_{10}^+ (t_* - t_{10}^+)}{e_{k2} f t_* (t_* - t_{10}^+) + e_1 B}$$

$$w = w_0^+ + \frac{D^+ (t - t_{10}^+)}{(t_{**} - t)} \quad (t_{10}^+ \leqslant t \leqslant t_{11}^+,\ A_1B_1,\ A_2B_2,\ KN) \tag{1.24}$$

$$w_0^+ = \frac{A (t_{10}^+ - t_1)}{t_* - t_{10}^+}, \qquad t_{11}^+ = t_{**} - \sqrt{\frac{D^+}{e_{k1}} \beta_2 t_{**} (t_{**} - t_{10}^+)}$$

$$D^+ = \frac{f\,[(e_2 - e_{k2}) \beta_2 (t_* - t_1) - e_{k2} e_2 (e_1 - e_{k1}) \beta (t_* - t_{10}^+)]}{e_2 \beta \beta_1 \beta_2 (t_* - t_{10}^+)}$$

$$w = w_1^+ + \frac{D_1^+ (t - t_{11}^+)}{t_* - t} \quad (t_{11}^+ \leqslant t \leqslant t_{11}^-,\ B_1E_1,\ A_2B_2,\ KN)$$

$$w_1^+ = w_0^+ + \frac{D^+ (t_{11}^+ - t_{10}^+)}{(t_{**} - t_{11}^+)}, \qquad t_{11}^- = p_1 + \sqrt{p_1^2 + q_1}$$

$$p_1 = \frac{f (h_2 - w_1^+) t_* + D_1^+ f t_{11}^+ + s_{11}}{f (h_2 - w_1^+ + D_1^+)}, \qquad q_1 = \frac{s_{11} t_*}{f (h_2 - w_1^+ + D_1^+)}$$

$$s_{11} = f t_{11}^+ (h_2 - w_1^+), \qquad D_1^+ = \frac{\mu_2 h_1 + (\mu_2 + \beta) w_1^+}{\mu_2 + \beta};$$

b₁) $t_{**} > t_{11}^-$

$$w = w_1^- + \frac{D_1^- (t - t_{11}^-)}{t_{**} - t} \quad (t_{11}^- \leqslant t < t_{**},\ E_1F_1,\ A_2B_2,\ KNP) \tag{1.25}$$

$$w_1^- = w_1^+ + \frac{D_1^+ (t_{11}^- - t_{11}^+)}{t_* - t_{11}^-}$$

$$D_1^- = \frac{\mu_2 h_1 - \mu_1 h_2 f + (\beta + \mu_2 + \mu_1 f) w_1^-}{\beta + \mu_2 + \mu_1 f};$$

b₂) $t_{**} \leqslant t_{11}^-,\ r_1 > 0$

$$w = w_1^- + \frac{D_1^- (t - t_{11}^-)}{(t_{**} - t)} \quad (t \leqslant t_{11}^-,\ E_1F_1,\ A_2B_2,\ KNQ) \tag{1.26}$$

$$r_1 = (t_* - t_{11}^-)(t_{**} - t_{11}^-) - \frac{(\mu_2 + \beta h_2)}{(\beta + \mu_2 + \mu_1 f)} \left[(t_* - t_{11}^-)^2 - \frac{(\mu_2 + \beta)}{(\mu_2 + \beta h_2)} D_1^+ t_* (t_* - t_{11}^+) \right];$$

b₃) $t_{**} < t_{11}^-,\quad r_1 \leqslant 0$

$$w = w_1^- + \frac{D_{1y}^- (t - t_{11}^-)}{t_{***} - t} \quad (t \leqslant t_{11}^-,\ E_1F_1,\ B_2E_2,\ KNQ) \tag{1.27}$$

$$D_{1y}^- = \frac{-\mu_1 h_1 f + (\beta + \mu_1 f) w_1^-}{\beta + \mu_1 f};$$

b₄) $t_{**} < t_{11}^-$

$$w = \frac{w_1^- (t_{Eu} - t_{11}^-)}{(t_{Eu} - t)} \quad (t \leqslant t_{11}^-,\ B_1E_1,\ B_2E_2,\ KNR); \tag{1.28}$$

c) $t_* > t_1,\quad t_{Eu} e_{k2} > t_1 e_2,\quad t_m > t_{10}^+,\quad e_{k1} \leqslant e_{k1}^*,\quad t_{**} > t_{10}^+$

$$w = w_0^+ + \frac{D_2^+ (t - t_{10}^+)}{t_{***} - t} \quad (t_{10}^+ \leqslant t \leqslant t_2,\ A_1B_1,\ B_2E_2,\ FU) \tag{1.29}$$

$$D_2^+ = \frac{-\mu_1 h_2 f + (\beta + \mu_1 f) w_0^+}{\beta + \mu_1 f}, \qquad t_2 = p_2 + \sqrt{p_2^2 + q_2}$$

$$p_2 = \frac{(h_1 + w_0^+) t_{***} - (h_1 + w_0^+ + D_2^+) t_{10}^+}{(h_1 + w_0^+ + D_2^+)}, \qquad q_2 = \frac{t_{10}^+ t_{***} (h_1 + w_0^+)}{(h_1 + w_0^+ + D_2^+)};$$

$c_1)\ t_{**} > t_2$

$$w = w_2 + \frac{D_2^-(t - t_2)}{t_{**} - t} \qquad (t_2 \leqslant t \leqslant t_{**},\ A_1B_1,\ E_2F_2,\ UL) \tag{1.30}$$

$$w_2 = w_0 + \frac{D_2^+(t_2 - t_{10}^+)}{t_{***} - t_2}, \qquad D_2^- = \frac{\mu_2 h_1 - \mu_1 h_2 f + (\beta + \mu_2 + \mu_1 f)\, w_2}{(\beta + \mu_2 + \mu_1 f)};$$

$c_2)\ t_{**} \leqslant t_2, \qquad g > 0$

$$w = w_2 + \frac{D_2^-(t - t_2)}{t_{**} - t} \qquad (t \leqslant t_2,\ A_1B_1,\ E_2F_2,\ UG) \tag{1.31}$$

$$g = (t_{***} - t_2)(t_{**} - t_2)(\beta + \mu_2 + \mu_1 f) - (\beta h_1 + \mu_1 f)(t_{***} - t_2)^2 + $$
$$+ D_2^+(\beta + \mu_1 f)\, t_{***}(t_{***} - t_{10}^+);$$

$c_3)\ t_{**} \leqslant t_2, \qquad g \leqslant 0$

$$w = w_2 + \frac{D_{2y}^-(t - t_2)}{(t_* - t)} \qquad (t \leqslant t_2,\ B_1E_1,\ E_2F_2,\ UG) \tag{1.32}$$

$$D_{2y}^- = \frac{\mu_2 h_1 + (\beta + \mu_2)\, w_2}{\beta + \mu_2};$$

$c_4)\ t_{**} \leqslant t_2$

$$w = \frac{w_2(t_{\mathrm{Eu}} - t_2)}{t_{\mathrm{Eu}} - t} \qquad (t \leqslant t_2,\ B_1E_1,\ B_2E_2,\ UV); \tag{1.33}$$

$d)\ t_* > t_1,\ t_{\mathrm{Eu}} e_{k2} > t_1 e_2, \quad t_m > t_{10}^+, \quad e_{k1} \leqslant e_{k1}^*, \quad t_{**} < t_{10}^+$ (1.34)

$$w = w_0^+ + \frac{D_2^+(t - t_{10}^+)}{t_{***} - t} \qquad (t \leqslant t_{10}^+,\ A_1B_1,\ B_2F_2,\ FT)$$

$$w = \frac{w_0^+(t_{\mathrm{Eu}} - t_{10}^+)}{(t_{\mathrm{Eu}} - t)} \qquad (t \leqslant t_{10}^+,\ A_1C_1,\ B_2E_2,\ FS).$$

For each of the cases (1.24–1.34) the load intervals and modes for which the respective solutions were obtained are given in parentheses, where curves representing the dependence of deflection on load are also shown. The curves for cases (a) and (b) are reproduced in Fig. 3 and for cases (c) and (d) in Fig. 4. The loads t_{10}^- and t_{11}^- in Fig. 3 coincide.

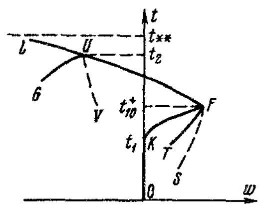

Fig. 4

The final conclusions are as follows. In the general case of a heterogeneous idealized column which retains its rectilinear unbuckled state in the elastic range, the deflection begins at the first critical load t_1 corresponding to the load at which the yield point is reached in one of the supporting segments; if the segment materials and geometrical parameters of the column are such that inequalities (1.4), $t_* < t_1$, and $t_{\mathrm{Eu}}\, e_{k2} \leq t_1 e_2$ are satisfied, the critical load t_1 determines the supporting capacity of the column. If, however, inequalities (1.4), (1.11), (1.13), and (1.20) or (1.4), (1.20), and (1.23) are satisfied, the supporting capacity of the column is determined by the critical load t_{10}^-. The supporting capacity of the column is determined by the critical load t_{11}^- in the case in which inequalities (1.4), (1.24), and (1.26) or (1.27) are valid, by the critical load t_2 when inequalities (1.4), (1.29), and (1.31) or (1.32) are satisfied, and by the critical load t_{10}^+ when inequalities (1.4) and (1.34) are satisfied.

If inequalities (1.4), (1.11), (1.13), and (1.18), or (1.4), (1.18), and (1.23), or (1.4), (1.24), and (1.25), or (1.4), (1.29), and (1.30) are satisfied the supporting capacity of the column can be expressed in terms of the load t_{**}.

2. The foregoing analysis shows that, in the general case, a heterogeneous idealized column begins to buckle at a load at which one of the segments passes into the plastic state. However, if certain limitations are imposed on the material characteristics, a column may remain in an unbuckled moment-free state even after the segments have passed into the plastic state. To determine these limitations, let us take in (1.8) $w = 0$, $k_i = 1$, $m_i = n_i = 0$ for several (at least two) close values of $t > t_1$. We then obtain

$$h_1\mu_2 - h_2\mu_1 f = 0, \quad \sigma_{01}\mu_1 - \sigma_{02}\mu_2 = 0. \tag{2.1}$$

From (2.1) follows

$$\frac{E_2}{E_1} = \frac{E_{k2}}{E_{k1}} = \frac{\sigma_{02}}{\sigma_{01}} = \alpha. \qquad (2.2)$$

Stress-strain diagrams of materials whose characteristics are related by Eq. (2.2)are called proportional [1]. Assuming that the diagrams of materials of the segments of a column are proportional from (1.8) we obtain the following expression for the critical load,

$$t_k = \frac{\omega^2 e_{k2}}{1 + \alpha f}.$$

Assuming that the materials and dimensions of the segments of a column are identical, we arrive at Shanley's load for an idealized column [2].

Thus, the usual formulation of the problem of stability of a column beyond the elastic limit can be used in cases in which the diagrams of the segment materials are proportional.

When the diagrams of the segment materials are different, for obtaining approximate estimates of the real critical loads it is useful to find the critical loads for columns made of materials with proportional diagrams [1].

If the real diagram of the material of the first segment is replaced by a diagram proportional to the diagram of the second segment (with a proportionality coefficient $\alpha = E_2/E_1$), we obtain the critical load

$$t_{k1} = t_k. \qquad (2.3)$$

Making the diagram of the real material of the second segment proportional to the diagram of the first segment with a coefficient α and determining the critical load, we obtain

$$t_{k2} = \frac{\omega^2 e_{k1} \alpha}{1 + \alpha f}. \qquad (2.4)$$

Comparing the second critical load t_{**} with critical loads (2.3) and (2.4), it is easy to show that

$$t_{k2} < t_{**} < t_{k1} \quad \text{at} \quad E_{k2}/E_{k1} > \alpha$$
$$t_{k1} < t_{**} < t_{k2} \quad \text{at} \quad E_{k2}/E_{k1} < \alpha.$$

These inequalities demonstrate the possibility of obtaining estimates of the critical loads with the aid of Shanley's loads for columns made of materials corresponding to proportionally reconstructed diagrams.

REFERENCES

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